On certain classes of algebraic curvature tensors

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Abstract

In this text we deal with Rakić duality principle. We search for a connection between Osserman and Rakić curvature tensor. We prove that 3-dimensional Rakić is Osserman. We investigate Rakić duality using Fiedler's skew-symmetric decomposition, and prove that Osserman curvature tensor with constant Fiedler's signs is Rakić.

1 Introduction

Let us begin with the basic notation and terminology which are used throughout this work. Let R be an algebraic curvature tensor on a vector space $\mathcal{V}$ equipped with an indefinite metric g of the signature $(\nu, n-\nu)$. The sign $\varepsilon_X = g(X,X)$ denotes the norm of $X \in \mathcal{V}$, and it determines various types of vectors. We say that $X \in \mathcal{V}$ is timelike (if $\varepsilon_X < 0$), spacelike ($\varepsilon_X > 0$), null ($\varepsilon_X = 0$), nonnull ($\varepsilon_X \neq 0$), or unit ($\varepsilon_X \in \{-1,1\}$). The curvature operator $\mathcal{R}$ is connected with R via equation $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W)$. If $(E_1, E_2, ..., E_n)$ is an orthonormal basis of $\mathcal{V}$, then we use short notations $\varepsilon_i = \varepsilon_{E_i}$ and $R_{ijkl} = R(E_i, E_j, E_k, E_l)$. For the initial definitions and deeper explanations of this topic, the reader can consult Gilkey's books [9, 10].

The Jacobi operator $\mathcal{J}_X : \mathcal{V} \to \mathcal{V}$ is a natural operator defined by $\mathcal{J}_X(Z) = \mathcal{R}(Z,X)(X)$ for all $Z \in \mathcal{V}$. In the case of nonnull $X \in \mathcal{V}$, $\mathcal{J}_X$ preserves nondegenerate hyperspace $\{X\}^\perp = \{Y \in \mathcal{V} : X \perp Y\}$, and we have the reduced Jacobi operator $\tilde{\mathcal{J}}_X : \{X\}^\perp \to \{X\}^\perp$, given by $\tilde{\mathcal{J}}_X = \mathcal{J}_X|_{\{X\}^\perp}$.

We say that R is an Osserman curvature tensor if the characteristic polynomial of $\mathcal{J}_X$ is constant on both pseudo-spheres, in particular on positive ($\varepsilon_X = 1$) and negative ($\varepsilon_X = -1$) one. In a pseudo-Riemannian setting, Jordan normal form plays a crucial role, since characteristic polynomial does not determine the eigen-structure of a symmetric linear operator. We say that R is a Jordan Osserman curvature tensor if the Jordan normal form of $\mathcal{J}_X$ is constant on both pseudo-spheres. An Osserman curvature tensor, whose Jacobi operator $\mathcal{J}_X$ is diagonalizable for all nonnull X, we call diagonalizable Osserman.

In the Riemannian setting ($\nu = 0$), it is known that a local two-point homogeneous space (flat or locally rank one symmetric space) has a constant

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characteristic polynomial on the unit sphere bundle. Osserman wondered if the converse held [15], and this question has been called the Osserman conjecture by subsequent authors. During the solution of some particular cases of the conjecture, the implication

$$\mathcal{J}_X(Y) = \lambda Y \Rightarrow \mathcal{J}_Y(X) = \lambda X \tag{1}$$

appeared naturally, and if it holds, it can significantly simplify some calculations. The first results in this topic are given by Chi [7], who proved the conjecture in the cases of dimensions $n \neq 4k$, $k > 1$. In his work he used the fact that (1) holds for extremal eigenvalues λ of the Jacobi operator. Rakić [16] proved the correctness of (1) for every eigenvalue λ of the Jacobi operator, and this statement has been called Rakić duality principle [9]. After that, Rakić duality is reproved by Gilkey [9], and it become a beneficial tool for the solution of the conjecture. Moreover, the best results in this topic gave Nikolayevsky [12, 13, 14], who used Rakić duality [13] to prove Osserman conjecture in all dimensions, except some possibilities in dimension $n = 16$.

The variant of the Osserman conjecture has been appeared in a pseudo-Riemannian setting. For example, in the Lorentzian setting ($\nu = 1$), an Osserman manifold necessarily has a constant sectional curvature [5]. Observation of Osserman manifolds in the signature $(2, 2)$ become very popular, and it is worth noting results from [6], which are based on the discussion of possible Jordan normal forms of the Jacobi operator.

This is why we start to investigate the duality principle for Osserman curvature tensor in a pseudo-Riemannian setting. In a pseudo-Riemannian setting, the implication (1) looks inaccurate, and therefore we corrected it in the following way [1, 4].

Definition 1 (Rakić duality) *We say that a curvature tensor R satisfies Rakić duality for the value λ, if for all mutually orthogonal units $X, Y \in \mathcal{V}$ holds*

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \Rightarrow \mathcal{J}_Y(X) = \varepsilon_Y \lambda X. \tag{2}$$

We say that R is Rakić if it satisfies Rakić duality for all $\lambda \in \mathbb{R}$.

The Rakić duality for Osserman curvature tensor works nicely for every known example, which motivated us to post the following conjecture.

Conjecture 1 *Osserman pseudo-Riemannian curvature tensor is Rakić.*

Unfortunately we failed to prove this conjecture in general. In our previous work we gave the affirmative answer only for the conditions of small index ($\nu \leq 1$) [1, 4], low dimension ($n \leq 4$) [1, 4, 3], and some possibilities with small numbers of eigenvalues of the reduced Jacobi operator [2]. Seeing that Rakić property is natural for Osserman curvature tensor (at least in the Riemannian setting), one can ask if the converse held.

Conjecture 2 *Rakić pseudo-Riemannian curvature tensor is Osserman.*

2 Three-dimensional case

In this section we deal with three-dimensional Rakić pseudo-Riemannian curvature tensor. Let us start with the following universal lemma.

Lemma 1 *If* $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$ *and* $\mathcal{J}_Y(X) = \varepsilon_Y \lambda X$*, then for all* $\alpha, \beta \in \mathbb{R}$ *holds* $\mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y) = \varepsilon_{\alpha X+\beta Y}\lambda(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)$.

Proof. This lemma is a consequence of the straightforward calculations.

$$\begin{aligned}
&\mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)\\
&\qquad = \mathcal{R}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y, \alpha X + \beta Y)(\alpha X + \beta Y)\\
&\qquad = \mathcal{R}(\varepsilon_Y \beta X, \beta Y)(\alpha X + \beta Y) + \mathcal{R}(-\varepsilon_X \alpha Y, \alpha X)(\alpha X + \beta Y)\\
&\qquad = -\varepsilon_Y \alpha \beta^2 \mathcal{J}_X(Y) + \varepsilon_Y \beta^3 \mathcal{J}_Y(X) - \varepsilon_X \alpha^3 \mathcal{J}_X(Y) + \varepsilon_X \alpha^2 \beta \mathcal{J}_Y(X)\\
&\qquad = \beta(\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)\mathcal{J}_Y(X) - \alpha(\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)\mathcal{J}_X(Y)\\
&\qquad = (\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)(\beta \varepsilon_Y \lambda X - \alpha \varepsilon_X \lambda Y)\\
&\qquad = \varepsilon_{\alpha X+\beta Y}\lambda(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)
\end{aligned}$$

□

Let us investigate Conjecture 2 for low dimension $n = 3$. It is known that three-dimensional Einstein (consequently it holds for Osserman) curvature tensor necessarily has constant sectional curvature.

Theorem 1 *Three-dimensional Rakić curvature tensor is of constant sectional curvature.*

Proof. In order to apply Rakić property we need to show that there is a pair (X, Y) of mutually orthogonal units, where Y is an eigenvector of $\mathcal{J}_X$. Let (E_1, E_2, E_3) be an arbitrary orthonormal basis of $\mathcal{V}$, such that $\varepsilon_1 = \varepsilon_2$. The matrix of the Jacobi operator $\mathcal{J}_{E_3}$ is

$$\mathcal{J}_{E_3} = \begin{pmatrix} \varepsilon_1 R_{1331} & \varepsilon_1 R_{2331} & 0 \\ \varepsilon_2 R_{1332} & \varepsilon_2 R_{2332} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore its reduced Jacobi operator $\tilde{\mathcal{J}}_{E_3}$ has characteristic polynomial

$$x^2 - (\varepsilon_1 R_{1331} + \varepsilon_2 R_{2332})x + \varepsilon_1\varepsilon_2 R_{1331}R_{2332} - \varepsilon_1\varepsilon_2 R_{2331}R_{1332} = 0.$$

Let D be a discriminant of the previous quadratic equation, then

$$D = (\varepsilon_1 R_{1331} + \varepsilon_2 R_{2332})^2 - 4\varepsilon_1\varepsilon_2 R_{1331}R_{2332} + 4\varepsilon_1\varepsilon_2 (R_{1332})^2.$$

Because of $\varepsilon_1 = \varepsilon_2$, we have $\varepsilon_1\varepsilon_2 = (\varepsilon_1)^2 = 1$, and thus

$$D = (\varepsilon_1 R_{1331} - \varepsilon_2 R_{2332})^2 + 4(R_{1332})^2 \geq 0.$$

If $D > 0$ then our quadratic equation has two distinct real roots, which represents two distinct eigenvalues of $\tilde{\mathcal{J}}_{E_3}$, and hence there are two (nondegenerate) one-dimensional eigenspaces. Otherwise, $D = 0$ gives $\varepsilon_1 R_{1331} = \varepsilon_2 R_{2332}$ and $R_{1332} = 0$. Thus $\tilde{\mathcal{J}}_{E_3}$ is diagonal with double root which is associated with a two-dimensional eigenspace.

The previous text proved that there exists an orthonormal basis (X, Y, Z), such that Y and Z are eigenvectors of $\mathcal{J}_X$. Consequently, by Rakić duality, X is an eigenvector of $\mathcal{J}_Y$. The conditions of Lemma 1 hold, and hence $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$ is an eigenvector of $\mathcal{J}_{\alpha X+\beta Y}$. Moreover, for $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2\varepsilon_X + \beta^2\varepsilon_Y \neq 0$, both $\alpha X + \beta Y$ and $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$ are mutually orthogonal and nonnull. The Jacobi operator is symmetric linear operator and therefore

$$g(\mathcal{J}_{\alpha X+\beta Y}(Z), \varepsilon_Y \beta X - \varepsilon_X \alpha Y) = g(Z, \mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)) = 0.$$

Hence $\mathcal{J}_{\alpha X+\beta Y}(Z)$ is orthogonal to both $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$ and $\alpha X + \beta Y$, which gives $\mathcal{J}_{\alpha X+\beta Y}(Z) \perp \mathrm{Span}\{X, Y\} = \mathrm{Span}\{Z\}^\perp$, and consequently Z is an eigenvector of $\mathcal{J}_{\alpha X+\beta Y}$. According to Rakić duality, $\alpha X + \beta Y$ is an eigenvector of $\mathcal{J}_Z$, which is possible only if $\mathrm{Span}\{X, Y\}$ is a two-dimensional eigenspace of $\tilde{\mathcal{J}}_Z$. Similarly one can prove that $\mathrm{Span}\{X, Z\}$ is an eigenspace of $\tilde{\mathcal{J}}_Y$. Thus arise

$$\frac{R(X, Y, Y, X)}{\varepsilon_X \varepsilon_Y} = \frac{R(Y, Z, Z, Y)}{\varepsilon_Y \varepsilon_Z} = \frac{R(X, Z, Z, X)}{\varepsilon_X \varepsilon_Z} = \kappa,$$

and $R(Y, X, X, Z) = R(X, Y, Y, Z) = R(X, Z, Z, Y) = 0$, which obviously completely describes R, and therefore R has constant sectional curvature κ. $\square$

3 Rakić duality and Fiedler's tensor

In this section we try to describe Rakić duality property. Let $(E_1, ..., E_n)$ be an arbitrary orthonormal basis of the vector space $\mathcal{V}$ of the signature $(\nu, n - \nu)$. Let us start with the left hand side of the equation (2),

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y. \tag{3}$$

The equation (3) means that Y is an eigenvector of $\mathcal{J}_X$ for the eigenvalue $\varepsilon_X \lambda$. The Jacobi operator can be expressed using the curvature tensor on the following way

$$\mathcal{J}_X(Y) = \mathcal{R}(Y, X)X = \sum_l \varepsilon_l R(Y, X, X, E_l)E_l.$$

If we set nonnull $X = \sum_i \alpha_i E_i$ and $Y = \sum_i \beta_i E_i$, the equation (3) become

$$\sum_{i,j,k,l} \varepsilon_l \beta_i \alpha_j \alpha_k R_{ijkl} E_l = \varepsilon_X \lambda \sum_l \beta_l E_l,$$

and finally

$$(\forall l) \sum_{i,j,k} \varepsilon_l \beta_i \alpha_j \alpha_k R_{ijkl} = \varepsilon_X \lambda \beta_l.$$

According to work of Fiedler [8], and later development by Gilkey [8, 9], for every algebraic curvature tensor R, there exist finite numbers of skew-symmetric tensors Ω of order 2 (i.e. the coordinates of Ω satisfy $\Omega_{ij} = -\Omega_{ji}$), such that R has a representation

$$R_{ijkl} = \sum_{\Omega} \varepsilon_{\Omega} \frac{1}{3} \left(2\Omega_{ij}\Omega_{kl} + \Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}\right),$$

with $\varepsilon_{\Omega} \in \{-1, 1\}$. Using this fact the equation (3) is equivalent to

$$(\forall l) \ \sum_{\Omega} \sum_{i,j,k} \frac{1}{3} \varepsilon_{\Omega}\varepsilon_l \beta_i \alpha_j \alpha_k \left(2\Omega_{ij}\Omega_{kl} + \Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}\right) = \varepsilon_X \lambda \beta_l.$$

We can simplify the previous formula using the symmetry by j and k.

$$\sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{ik}\Omega_{jl} = \sum_{i,k,j} \beta_i \alpha_k \alpha_j \Omega_{ij}\Omega_{kl} = \sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{ij}\Omega_{kl}$$

$$\sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{il}\Omega_{jk} = \sum_{i,k,j} \beta_i \alpha_k \alpha_j \Omega_{il}\Omega_{kj} = -\sum_{i,j,k} \beta_i \alpha_j \alpha_k \Omega_{il}\Omega_{jk} = 0$$

Therefore the equation (3) is equivalent to

$$(\forall l) \ \sum_{\Omega} \sum_{i,j,k} \varepsilon_{\Omega}\varepsilon_l \beta_i \alpha_j \alpha_k \Omega_{ij}\Omega_{kl} = \varepsilon_X \lambda \beta_l.$$

Let us split sums on the left hand side

$$(\forall l) \ \sum_{\Omega} \varepsilon_{\Omega}\varepsilon_l \sum_{i,j} \alpha_i \beta_j \Omega_{ij} \sum_{k} \alpha_k \Omega_{kl} = -\varepsilon_X \lambda \beta_l.$$

If we introduce the short notation

$$\Theta^{\Omega}_{PQ} = \sum_{i,j} \mu_i \nu_j \Omega_{ij},$$

for $P = \sum_i \mu_i E_i$ and $Q = \sum_j \nu_j E_j$, the equation (3) become equivalent to

$$(\forall l) \ \sum_{\Omega} \varepsilon_{\Omega} \Theta^{\Omega}_{XY} \Theta^{\Omega}_{XE_l} = -\varepsilon_l \varepsilon_X \lambda \beta_l. \tag{4}$$

Using (3) $\Leftrightarrow$ (4) and $\Theta^{\Omega}_{YX} = -\Theta^{\Omega}_{XY}$, we get the equivalent form of Rakić duality condition (2)

$$(\forall l) \ \sum_{\Omega} \varepsilon_{\Omega} \Theta^{\Omega}_{XY} \Theta^{\Omega}_{XE_l} = -\varepsilon_l \varepsilon_X \lambda \beta_l \ \Rightarrow \ (\forall l) \ \sum_{\Omega} \varepsilon_{\Omega} \Theta^{\Omega}_{XY} \Theta^{\Omega}_{YE_l} = \varepsilon_l \varepsilon_Y \lambda \alpha_l. \tag{5}$$

Let us sum all n equations ($1 \leq l \leq n$) from (4) multiplied by β_l

$$\sum_{l} \beta_l \sum_{\Omega} \varepsilon_{\Omega} \Theta^{\Omega}_{XY} \Theta^{\Omega}_{XE_l} = -\sum_{l} \beta_l \varepsilon_l \varepsilon_X \lambda \beta_l,$$

After the substitutions $\sum_l \beta_l \Theta^{\Omega}_{XE_l} = \Theta^{\Omega}_{XY}$ and $\sum_l \varepsilon_l \beta_l^2 = \varepsilon_Y$ we get important equation.

$$\sum_{\Omega} \varepsilon_{\Omega} (\Theta^{\Omega}_{XY})^2 = -\varepsilon_X \varepsilon_Y \lambda. \tag{6}$$

Let us stop here to notice the following interesting statements in the case when Fiedler's terms have a constant signs, i.e. $\varepsilon_{\Omega} = \mathrm{const}$.

Theorem 2 *If R is a curvature tensor with $\varepsilon_{\Omega} = \mathrm{const}$, then it satisfies Rakić duality for the value 0.*

Proof. If the sign ε_{Ω} is constant for all skew-symmetric tensors Ω in the sense of Fiedler, then the equation (6) gives

$$0 \leq \sum_{\Omega} (\Theta^{\Omega}_{XY})^2 = -\varepsilon_{\Omega} \varepsilon_X \varepsilon_Y \lambda.$$

The value $\lambda = 0$ gives $\sum_{\Omega} (\Theta^{\Omega}_{XY})^2 = 0$ and therefore $\Theta^{\Omega}_{XY} = 0$ for all Ω. The formula (5) obviously holds and R satisfies Rakić duality for the value 0. $\square$

Theorem 3 *A diagonalizable Osserman curvature tensor R with $\varepsilon_{\Omega} = \mathrm{const}$ is Rakić.*

Proof. Like the previous proof, the equation (6) with $\varepsilon_{\Omega} = \mathrm{const}$ gives $0 \leq -\varepsilon_{\Omega} \varepsilon_X \varepsilon_Y \lambda$. Since by Theorem 2, R satisfies Rakić duality for the value 0, we set $\lambda \neq 0$, and therefore $\varepsilon_{\Omega} \varepsilon_X \varepsilon_Y \lambda < 0$. It implies the constant sign of ε_Y, which proves that an eigenspace of $\mathcal{J}_X$ for an eigenvalue $\varepsilon_X \lambda$ has the same type of vectors. Especially, there are no nonzero null vectors in the eigenspace of $\mathcal{J}_X$ for an eigenvalue $\varepsilon_X \lambda$. According to our previous work [1, 4], diagonalizable Osserman curvature tensor, such that $\mathcal{J}_X$ has no null eigenvector for an eigenvalue $\varepsilon_X \lambda$, satisfies the duality principle for the value λ, which completes the proof. $\square$

The diagonalizability from Theorem 3 is a natural condition. Let us remark that, according to Gilkey and Ivanova [11], Jordan Osserman curvature tensor of a non-balanced signature ($n \neq 2\nu$) is necessarily diagonalizable.

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